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Regarding the Hadwiger Conjecture

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ABSTRACT
The *Hadwiger conjecture* (see [1] or [2]) is well known. The *Hadwiger conjecture* states that every graph G satisfies $\chi(G) \leq \eta(G)$ [where $\eta(G)$ is the *hadwiger number* of G (i.e. the maximum of p such that G is *contractible* to the complete graph K_p), and $\chi(G)$ is the chromatic number of G. We recall (see [2]) that the famous four-color problem is a special case of the Hadwiger conjecture]. In this paper, we give the original reformulation of the Hadwiger conjecture and the algebraic reformulation of the Hadwiger conjecture. The algebraic reformulation of the Hadwiger conjecture (which is based on the original reformulation of the Hadwiger conjecture) shows that the proof of this conjecture is strongly linked to a very small class of graphs.

Keywords: true pal, hadwiger index, parent, optimal coloration, uniform graph, relative subgraph, hadwigerian, hadwigerian subgraph, maximal hadwigerian subgraph, hadwiger caliber

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PRELIMINARIES

Recall that in a graph $G = [V(G), E(G), \chi(G), \omega(G), \alpha(G), \eta(G)]$, $V(G)$ is the set of vertices, $E(G)$ is the set of edges, $\chi(G)$ is the *chromatic number* $\Big[$ i.e. the smallest number of colors needed to color all vertices of G such that two adjacent vertices do not receive the same color $\Big]$, $\omega(G)$ is the *clique number* of G $\Big[$i.e. the size of a largest *clique* of G. Recall that a graph F is a *subgraph* of G, if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. We say that a graph F is an *induced subgraph* of G by Z, if F is a subgraph of G such that $V(F) = Z$, $Z \subseteq V(G)$, and for any pair of vertices x and y of F (note that x and y are in $V(F) = Z$), xy is an edge of F if and only if xy is an edge of G. For $X \subseteq V(G)$, $G \setminus X$ denotes the *subgraph* of G *induced* by $V(G) \setminus X$. A *clique* of G is a subgraph of G that is *complete*; such a subgraph is necessarily an induced subgraph (recall that a graph K is complete if every pair of vertices of K is an edge of K) $\Big]$, $\alpha(G)$ is the *stability number* of G $\Big[$ i.e. the size of a largest *stable set* of G. Recall that a *stable set* of a graph G is a set of vertices of G that induces a subgraph with no edges $\Big]$; $\eta(G)$ is the *hadwiger number* of G and is the maximum of p such that G is *contractible* to the complete graph K_p $\Big[$ recall that, if e is an edge of G incident to x and y, we can obtain a new graph from G by removing the edge e and identifying x and y so that the resulting vertex is incident to all those edges (other than e) originally incident to x or to y. This is called *contracting* the edge e. If a graph F can be obtained from G by a succession of such edge-contractions, then, G is *contractible* to F. The maximum of p such that G is contractible to the complete graph K_p is the *hadwiger number* of G, and is denoted by $\eta(G)$ $\Big]$. The Hadwiger conjecture states that every graph G is $\eta(G)$ colorable $\Big[$ i.e. we can color all vertices of G with $\eta(G)$ colors such that two adjacent vertices do not receive the same color $\Big]$. In this paper, we introduce some definitions that are not standard in the literature of Graph Theory, and, using these non-standard definitions coupled with some properties, we give the original reformulation of the Hadwiger conjecture and the algebraic reformulation of the Hadwiger conjecture. The algebraic reformulation of the Hadwiger conjecture (which is based on the original reformulation of the Hadwiger conjecture) shows that the proof of this conjecture is strongly linked to a very small class of graphs. That being said, all results of this paper are original, and therefore, are not related to strong investigations that have been done on the Hadwiger conjecture in the past by other authors. It is easy to see.

Assertion 0.0. *Let G be a graph and F be a subgraph of G. Then $\omega(G) \leq \chi(G)$ and $\eta(F) \leq \eta(G)$.* $\square$

It is very easy to prove that:

Assertion 0.1. *The Hadwiger conjecture is true for every graph G such that $0 \leq \chi(G) \leq 2$.* $\square$

That being said, this paper is divided into two sections. In Section.*1*, we recall standard definitions known in Graph Theory, and we define the graph parameter denoted by τ $\Big[$ the graph parameter τ is called the *hadwiger index* $\Big]$, and, via the parameter τ, we give *the original reformulation of the Hadwiger conjecture*. This original reformulation is simple and crucial for the algebraic reformulation of the Hadwiger conjecture. In Section.*2*, we use the original reformulation of the *Hadwiger conjecture* to introduce *uniform* graphs and *relative* subgraphs $\Big[$ uniform graphs and relative subgraphs are crucial for the algebraic reformulation of the Hadwiger conjecture $\Big]$, and we give some elementary properties of these graphs; in Section.*2*, we also define another graph parameter denoted by a $\Big[$ the graph parameter a is called the *hadwiger caliber*, and is related to the hadwiger index τ defined in Section.*1* $\Big]$, and using the graph parameter a, we give the algebraic reformulation of the *Hadwiger conjecture*. The algebraic reformulation of the Hadwiger conjecture (which is based on the original reformulation of the Hadwiger conjecture) shows that the proof of this conjecture is strongly linked to a very small class of graphs called *uniform* graphs. Here, every graph is finite, is simple and undirected.

1. *THE HADWIGER INDEX OF A GRAPH AND THE ORIGINAL REFORMULATION OF THE HADWIGER CONJECTURE*

In this section, we introduce some important definitions that are not standard. In particular, we define a graph parameter called the *hadwiger index* [and denoted by τ], and we use it to give the original reformulation of the Hadwiger conjecture.

Definition 1.0. (*true pal*). We say that a graph G is a *true pal* of a graph F, if F is a subgraph of G and $\chi(F) = \chi(G)$. *trpl(F)* denotes the set of all true pals of F; **so** $G \in trpl(F)$ **means** G is a true pal of F .

Definition 1.1. (*complete $\omega(Q)$-partite* graph and Ω). We recall that a graph Q is a *complete $\omega(Q)$-partite* graph, if there exists a partition $\Xi(Q) = \{Y_1, ..., Y_{\omega(Q)}\}$ of $V(Q)$ into $\omega(Q)$ stable set(s), such that $x \in Y_j \in \Xi(Q)$, $y \in Y_k \in \Xi(Q)$ and $j \neq k$, $\Rightarrow$ x and y are adjacent in Q. Ω denotes the set of all *complete $\omega(Q)$-partite graphs*; **so** $Q \in \Omega$ **means** Q is a complete $\omega(Q)$-partite graph. For example, if G is a complete $\omega(G)$-partite graph with $\omega(G) \in \{0, 1, 2, 3, 4, ..., etc..\}$, then $G \in \Omega$. *More generally*, G is a complete $\omega(G)$-partite graph with $\omega(G) \geq 0$, *if and only if* , $G \in \Omega$. It is immediate that

$\chi(Q) = \omega(Q)$ for all $Q \in \Omega$ $\Big[$ it is also immediate that, for every $Q \in \Omega$, the partition $\Xi(Q) = \{Y_1, ..., Y_{\omega(Q)}\}$ of $V(Q)$ into $\omega(Q)$ stable set(s) is *canonical* $\Big]$.

Now, using the previous definitions, then the following Assertion becomes immediate.

Assertion 1.2. *Let G be a graph. Then, there exists a graph $P \in \Omega$ such that P is a true pal of G [i.e. there exists $P \in \Omega$ such that $P \in trpl(G)$].*

Proof. Indeed, let G be a graph and let $\Xi(G) = \{Y_1, ..., Y_{\chi(G)}\}$ be a partition of $V(G)$ into $\chi(G)$ stable set(s) [it is immediate that such a partition $\Xi(G)$ exists]. Now let Q be a graph defined as follows: (i) $V(Q) = V(G)$, (ii) $\Xi(Q) = \{Y_1, ..., Y_{\chi(G)}\} = \Xi(G)$ is a partition of $V(Q)$ into $\chi(G)$ stable set(s) such that $x \in Y_j \in \Xi(Q)$, $y \in Y_k \in \Xi(Q)$ and $j \neq k$, $\Rightarrow$ x and y are adjacent in Q. Clearly $Q \in \Omega$, $\chi(Q) = \omega(Q) = \chi(G)$, and G is visibly a subgraph of Q; observe that Q is a true pal of G such that $Q \in \Omega$ [because G is a subgraph of Q and $\chi(Q) = \chi(G)$ and $Q \in \Omega$]. Now put $Q = P$; Assertion 1.2 follows.□

Using Assertion 1.2, let us define.

Definition 1.3. (*parent*). We say that a graph P is a *parent* of a graph F, if $P \in \Omega \bigcap trpl(F)$. In other words, a graph P is a *parent* of F, if P is a complete $\omega(P)$-partite graph and P is also a true pal of F [note that such a P clearly exists, via Assertion 1.2]. *parent(F)* denotes the set of all parents of F; **so** $P \in parent(F)$ **means** P is a parent of F.

The following assertion is an immediate consequence of Definition 1.3 and Assertion 1.2.

Assertion 1.4. *Let G be a graph. Then, there exists a graph P which is a parent of G [i.e. there exists a graph P such that $P \in parent(G)$].*

Proof. Immediate [use Definition 1.3 and Assertion 1.2]. □.

Using the definition of a parent [use Definition 1.3], the definition of a true pal [use Definition 1.0], the definition of Ω [use Definition 1.1], and the definition of $\eta(G)$ (use **Preliminary**), then the following two assertions are immediate.

Assertion 1.5. *Let G be a graph. Then, there exists a graph S such that G is a true pal of S and $\eta(S)$ is minimum for this property.* □

Assertion 1.6. *Let F be a graph and let $P \in parent(F)$; then $\chi(F) = \chi(P) = \omega(P)$.* □

Now, we define the *hadwiger index* and a *son*.

Definitions 1.7. *(the hadwiger index and a son).* Let G be a graph and put $\mathcal{A}(G) = \{H; G \in trpl(H)\}$ [clearly $\mathcal{A}(G)$ is the set of all graphs H, such that G is a true pal of H]. Then the *hadwiger index* of G is denoted by $\tau(G)$, where $\tau(G) = \min_{F \in \mathcal{A}(G)} \eta(F)$; and *a son of* G is a graph S such that $S \in \mathcal{A}(G)$ and $\eta(S) = \tau(G)$ [using Assertion 1.5, then it becomes immediate to see that for every graph G, $\tau(G)$ exists and is well defined. Moreover, it is also immediate to see that a son S of G exists and is not necessarily unique].

Now using Definitions 1.7, then the following Proposition is immediate.

Proposition 1.7'. *Let (K, G, F, P), where K is a complete graph, $G \in \Omega$, F is a graph and $P \in parent(F)$. We have the following elementary properties.*
$(1.7'.1)$. If $\omega(G) \leq 1$, then $\omega(G) = \chi(G) = \eta(G) = \tau(G)$.
$(1.7'.2)$. $\omega(K) = \chi(K) = \eta(K) = \tau(K)$.
$(1.7'.3)$. $\omega(G) \geq \tau(G)$.
$(1.7'.4)$. If $G \in trpl(F)$, then $\tau(G) \leq \tau(F)$.
$(1.7'.5)$. $\tau(P) \leq \tau(F)$. $\square$
Proof. Properties $(1.7'.1)$ and $(1.7'.2)$ are immediate. Property $(1.7'.3)$ is very easy [indeed recall that $G \in \Omega$, and clearly $\chi(G) = \omega(G)$. Now put $\mathcal{A}(G) = \{H; G \in trpl(H)\}$ and let K' be a complete graph such that $\omega(K') = \omega(G)$ and $V(K') \subseteq V(G)$; clearly K' is a subgraph of G and

$$\chi(G) = \omega(G) = \chi(K') = \omega(K') = \eta(K') = \tau(K') \qquad (1.0).$$

In particular K' is a subgraph of G with $\chi(G) = \chi(K')$, and therefore, G is a true pale of K'; so $K' \in \mathcal{A}(G)$ and clearly

$$\tau(G) \leq \eta(K') \qquad (1.1).$$

Observe that $\omega(G) = \eta(K')$ [use *(1.0)*] and inequality *(1.1)* immediately becomes $\tau(G) \leq \omega(G)$. Property $(1.7'.3)$ follows]. Property $(1.7'.4)$ is obvious [indeed put $\mathcal{A}(G) = \{H; G \in trpl(H)\}$ and let S be a son F; recalling that $G \in trpl(F)$, clearly $G \in trpl(S)$, so $S \in \mathcal{A}(G)$ and clearly

$$\tau(G) \leq \eta(S) \qquad (1.2).$$

Now observe that $\eta(S) = \tau(F)$ [because S is a son of F] and inequality *(1.2)* immediatly becomes $\tau(G) \leq \tau(F)$. Property $(1.7'.4)$ follows]. Property $(1.7'.5)$ is immediate [indeed observe that $P \in parent(F)$ (in particular $P \in trp(F)$) and use Property $(1.7'.4)$]. Proposition $1.7'$ follows.$\square$

Now the following Theorem is the original reformulation of the Hadwiger conjecture.

Theorem 1.8. (The original reformulation of the Hadwiger conjecture). *The following are equivalent.*
(1) The Hadwiger conjecture is true (i.e. For every graph H, we have $\chi(H) \leq \eta(H)$).
(2) For every graph F, we have $\chi(F) \leq \tau(F)$.
(3) For every $G \in \Omega$, we have $\omega(G) = \tau(G)$.
Proof. (1) $\Rightarrow$ (2)]. Indeed let F be a graph and let S be a son of F, clearly $\chi(S) \leq \eta(S)$; now observing that $\chi(S) = \chi(F)$ [since $F \in trpl(S)$, by using the definition of a son S of F] and $\eta(S) = \tau(F)$ [because S is a son of F], then the previous inequality immediately becomes $\chi(F) \leq \tau(F)$.
(2) $\Rightarrow$ (3)]. Let $G \in \Omega$; in particular G is a graph and so $\chi(G) \leq \tau(G)$. Note $\chi(G) = \omega(G)$ [since $G \in \Omega$] and the previous inequality becomes

$$\omega(G) \leq \tau(G) \qquad (1.3).$$

Now observing that

$$\omega(G) \geq \tau(G) \qquad (1.4)$$

[remark that $G \in \Omega$ and use property $(1.7'.3)$ of Proposition $1.7'$], clearly $\omega(G) = \tau(G)$ [use *(1.3)* and *(1.4)*].

(3) $\Rightarrow$ (1)]. Let H be a graph and let $P \in parent(H)$, then

$$\tau(P) \leq \tau(H) \qquad (1.5)$$

[use property (1.7′.5) of Proposition 1.7′]. Observe $P \in \Omega$ [since $P \in parent(H)$] so $\omega(P) = \tau(P)$ [because $P \in \Omega$] and $\chi(H) = \chi(P) = \omega(P)$ [since $P \in parent(H)$]; clearly $\tau(P) = \chi(H)$ (use the previous) and inequality (1.5) becomes

$$\chi(H) \leq \tau(H) \qquad (1.6).$$

Since it is immediate that

$$\tau(H) \leq \eta(H) \qquad (1.7),$$

clearly

$$\chi(H) \leq \tau(H) \leq \eta(H) \qquad (1.8)$$

[use *(1.6)* and *(1.7)*] and so $\chi(H) \leq \eta(H)$ [use *(1.8)*]. □

We will use Theorem 1.8 in Section.*2* to define uniform graphs which are crucial for the algebraic reformulation of the Hadwiger conjecture. Now using the definition of the hadwiger number η (use *Preliminary*) and the definition of the hadwiger index τ (use Definitions 1.7) and Assertion 0.1, then the following assertion is trivial and we leave it to the reader.

Assertion 1.9. *We have the following three properties.*
(1). The Hadwiger conjecture is true for every graph G' *such that* $0 \leq \chi(G') \leq 2$ *[i.e. For every graph* H *such that* $0 \leq \chi(H) \leq 2$*, we have* $\chi(H) \leq \eta(H)$ *].*
(2) For every graph F *such that* $0 \leq \chi(F) \leq 2$*, we have* $\chi(F) \leq \tau(F)$*.*
(3) For every $G \in \Omega$ *such that* $0 \leq \chi(G) \leq 2$*, we have* $\omega(G) = \tau(G)$*.* □

2. *UNIFORM GRAPHS, RELATIVE SUBGRAPHS, THE HADWIGER CALIBER AND SOME CONSEQUENCES : THE ALGEBRAIC REFORMULATION OF THE HADWIGER CONJECTURE*

In this section, we use the original reformulation of the *Hadwiger conjecture* given by Theorem 1.8 to introduce *uniform* graphs and *relative* subgraphs [uniform graphs and relative subgraphs are crucial for the algebraic reformulation of the Hadwiger conjecture], and we give some elementary properties of these graphs. In this section, we also define another graph parameter denoted by a [the graph parameter a is called the *hadwiger caliber*, and is related to the hadwiger index defined in Section.*1*], and using the graph parameter a coupled with some properties, we give the algebraic reformuation of the *Hadwiger conjecture*. The algebraic reformulation of the Hadwiger conjecture is based on the original reformulation of the Hadwiger conjecture and shows that the proof of this conjecture is strongly linked to a very small class of graphs mentioned above and called *uniform* graphs. In this section, the definition of *true pal* (use Definition 1.0), the denotation of Ω (use Definition 1.1), the definition of *parent* (use Definition 1.3), the definition of the *hadwiger index* τ (use Definitions 1.7), and the definition of the hadwiger number η (use *preliminary*), are fundamental and crucial. Now let us remark.

Remark 2.0. *Let* F *be a graph and let* P *be a parent of* F*; then* $\tau(P) \leq \tau(F)$*.*
Proof. Immediate and is an obvious consequence of property (1.7′.5) of Proposition 1.7′.□

Remark 2.1. *Let K be a complete graph; then $\tau(K) = \omega(K) = \chi(K) = \eta(K)$.*
Proof. Immediate and is an obvious consequence of property $(1.7'.2)$ of Proposition $1.7'$. □

Inspirated by Theorem 1.8, we are going to define a new class of graphs in Ω [called *uniform graphs*]; we will also define *relative subgraphs*, and we will present some properties related to these graphs. These properties are elementary and curiously, are crucial for the algebraic reformulation of the Hadwiger conjecture. Before, let us define.

Definition 2.2. [*optimal coloration and* $\ominus(G)$]. An *optimal coloration* of a graph G is a partition $\Xi(G) = \{Y_1, ..., Y_{\chi(G)}\}$ of $V(G)$ into $\chi(G)$ stable set(s) [where $\chi(G)$ is the chromatic number of G]; $\ominus(G)$ denotes the set of all optimal colorations of G; **so**, $\Xi(G) \in \ominus(G)$ **means** $\Xi(G)$ is an optimal coloration of G.

Definition 2.3. [**The** *canonical coloration*]. Let G be a graph and let $\Xi(G) \in \ominus(G)$. We say that $\Xi(G)$ is **the** *canonical coloration* of G, **if and only if**, $\ominus(G) = \{\Xi(G)\}$ [observe that such a canonical coloration does not always exist].

Using the denotation of $\ominus(G)$ [use Definition 2.2], then the following Assertion is immediate.

Assertion 2.4. *Let $G \in \Omega$ and let $\Xi(G) \in \ominus(G)$. Then $\Xi(G)$ is* **the** *canonical coloration of G* [*i.e.* $\ominus(G) = \{\Xi(G)\}$, *by Definition* 2.3] .
Proof. Immediate, by observing that $G \in \Omega$. □

So, let $G \in \Omega$ and let $\Xi(G) \in \ominus(G)$; then Assertion 2.4 clearly says that $\Xi(G)$ is **the** canonical coloration of G [indeed, we have no choice, since $\ominus(G) = \{\Xi(G)\}$].

Definition 2.5. *[Uniform graph].* Let $G \in \Omega$ and let $\Xi(G)$ be the canonical coloration of G [observe that $\Xi(G)$ exists via Assertion 2.4]; we say that G is *uniform*, **if** for every $Y \in \Xi(G)$, we have $card(Y) = \alpha(G)$, where $card(Y)$ is the cardinality of Y and $\alpha(G)$ is the *stability number* of G.

Definition 2.5 gets sense, since $G \in \Omega$ and so $\Xi(G)$ is canonical [via Assertion 2.4]. Using the definition of a uniform graph [use Definition 2.5], then the following Assertion is immediate.

Assertion 2.6. *Let $G \in \Omega$ and let $\Xi(G)$ be the canonical coloration of G [observe that $\Xi(G)$ exists via Assertion 2.4]. We have the following trivial properties.*
(2.6.0). If $0 \leq \omega(G) \leq 1$, then G is uniform.
(2.6.1). If $0 \leq \alpha(G) \leq 1$, then G is uniform.
(2.6.2). If G is a complete graph, then G is uniform.
(2.6.3). If $\alpha(G) \geq 2$ and if for every $Y \in \Xi(G)$ we have $card(Y) = \alpha(G)$, then G is uniform and is not a complete graph.
Proof. Properties (2.6.0) and (2.6.1) are trivial [it suffices to use Definition 2.5]; property (2.6.2) is an immediate consequence of property (2.6.1); and property (2.6.3) is trivial $\Big[$ indeed, observe (by the hypotheses) that $G \in \Omega$ and use Definition 2.5 $\Big]$.□

Uniform graphs have nice properties when we study isomorphism of graphs.

Recall 2.7. Recall that two graphs are *isomorphic* if there exists a one to one correspondence between their vertex set that preserves adjacency.

Assertion 2.8. *Let $G \in \Omega$; then there exists a uniform graph U which is isomorphic to a parent of G [use Definition 1.3 for the meaning of parent].*
Proof. **If** $0 \leq \omega(G) \leq 1$, clearly G is uniform [use property (2.6.0) of Assertion 2.6]; now put $U = G$, clearly U is a uniform graph which is a parent of G. Now, **if** $\omega(G) \geq 2$, let $\Xi(G)$ be the canonical coloration of G [observe that $\Xi(G)$ exists, by remarking that $G \in \Omega$ and by using Assertion 2.4]; since it is immediate that $\chi(G) = \omega(G)$, clearly $\Xi(G)$ is of the form $\Xi(G) = \{Y_1, ..., Y_{\omega(G)}\}$. Now let Q be a graph defined as follows: (i) $\Xi(Q) = \{Z_1, ..., Z_{\omega(G)}\}$ is a partition of $V(Q)$ into $\omega(G)$ stable sets such that, $x \in Z_j \in \Xi(Q)$, $y \in Z_k \in \Xi(Q)$ and $j \neq k$, $\Rightarrow$ x and y are adjacent in Q; (ii) For every $j = 1, 2, ..., \omega(G)$ and for every $Z_j \in \Xi(Q) = \{Z_1, ..., Z_{\omega(G)}\}$, $card(Z_j) = \alpha(G)$. Clearly $Q \in \Omega$, $card(V(Q)) = \omega(G)\alpha(G)$, Q is uniform, $\chi(Q) = \omega(Q) = \omega(G) = \chi(G)$, and visibly, G is isomorphic to a subgraph of Q; observe that Q is isormorphic to a true pal of G and $Q \in \Omega$ [because G is isomorphic to a subgraph of Q and $\chi(Q) = \omega(Q) = \omega(G) = \chi(G)$ and $Q \in \Omega$] and Q is uniform. Using the previous and the definition of a parent, then we immediately deduce that Q is a uniform graph which is isomorphic to a parent of G. Now put $Q = U$; clearly U is a uniform graph which is isomorphic to a parent of G. Assertion 2.8 follows.□

Now the following assertion is only an immediate consequence of Assertion 2.8.

Assertion 2.9. *Let H be a graph [H is not necessarily in Ω]; then there exists a uniform graph U which is isomorphic to a parent of H.*
Proof. Let P be a parent of H [such a P exists via Assertion 1.4] and let U be a uniform graph such that U is isomorphic to a parent of P [such a U exists, by observing that $P \in \Omega$ and by using Assertion 2.8]; clearly U is a uniform graph and is isomorphic to a parent of H [since U is a uniform graph which is isomorphic to a parent of P and P is a parent of H] .□

Now we define *relative* subgraphs.

Definition 2.10. *(relative subgraph).* Let G and F be uniform. Now let $\Xi(G)$ be the canonical coloration of G and let $\Xi(F)$ be the canonical coloration of F [observe that the couple $(\Xi(G), \Xi(F))$ exists, by remarking that $G \in \Omega$ and $F \in \Omega$, and by using Assertion 2.4]. We say that F is a *relative subgraph* of G, **if** $\Xi(F) \subseteq \Xi(G)$ $\Big[$ it is immediate that the previous gets sense, since in particular $(G, F) \in \Omega \times \Omega$ (because G and F are uniform), and so $\Xi(G)$ and $\Xi(F)$ are canonical (via Assertion 2.4 and Definition 2.3). It is also immediate that relative subgraphs are defined for uniform graphs, and only for uniform graphs $\Big]$.

Using the definition of a relative subgraph [use Definition 2.10] and the definition of uniform graph [use Definition 2.5], then the following assertion is immediate and will help us later.

Assertion 2.11. *Let (P, U) be a couple of uniform graphs such that $\omega(P) \geq 1$ and $\omega(U) \geq 1$. Now let $\Xi(P)$ be the canonical coloration of P and let $\Xi(U)$ be the canonical coloration of U [observe that the couple $(\Xi(P), \Xi(U))$ exists, by remarking that $P \in \Omega$ and $U \in \Omega$, and by using Assertion 2.4]. Then we have the following trivial properties.*
(2.11.0). If U is a relative subgraph of P, then $\alpha(U) = \alpha(P)$ and $\omega(U) \leq \omega(P)$.
(2.11.1). If U is a relative subgraph of P and if $\omega(U) = \omega(P)$, then $U = P$.

(2.11.2). If U is a relative subgraph of P and if $\omega(U) < \omega(P)$, then there exists $Y \in \Xi(P)$ such that U is a relative subgraph of $P \setminus Y$.
(2.11.3). If $\alpha(U) = \alpha(P)$ and $\omega(U) = \omega(P)$, then U and P are isomorphic.
(2.11.4). If $\omega(P) \geq 2$, then, for every $Y \in \Xi(P)$, $P \setminus Y$ is a relative subgraph of P and $P \setminus Y$ is uniform and $\omega(P \setminus Y) = \omega(P) - 1$ and $\alpha(P \setminus Y) = \alpha(P)$.
Proof. Properties (2.11.0) and (2.11.1) and (2.11.2) are immediate [it suffices to use the definition of a uniform graph and the definition of a relative subgraph]. Properties (2.11.3) and (2.11.4) are trivial consequences of the definition of uniform graphs and relative subgraphs. □

Now we introduce again definitions that are not standard; in particular, we introduce a graph parameter denoted by a and called the *hadwiger caliber* [the *hadwiger caliber* a is related to the *hadwiger index* τ introduced in Definitions 1.7 (Section.**1**)]. Before, let us define.

Definition 2.12. *(Fundamental).* We say that a graph G is *hadwigerian*, **if** G is uniform and **if** $\omega(G) = \tau(G)$ [use Definitions 1.7 for the meaning of $\tau(G)$ and Definition 2.5 for the meaning of uniform].

The following two assertions are obvious consequences of Remark 2.1 and Definition 2.12.

Assertion 2.13. *Let K be a complete graph; then K is hadwigerian.*
Proof. Immediate, and is a consequence of Remark 2.1 and Definition 2.12 and property (2.6.2) of Assertion 2.6.□

Assertion 2.14 *The set of all complete graphs is an obvious example of hadwigerian graphs.*
Proof. Immediate, and is a trivial consequence of Assertion 2.13.□

Definitions 2.15. (*hadwigerian subgraph* and *maximal hadwigerian subgraph*). Let G be uniform. We say that a graph F is a *hadwigerian subgraph* of G, if F is *hadwigerian* and is a relative subgraph of G [use Definition 2.10 for the meaning of a relative subgraph and Definition 2.12 for the meaning of *hadwigerian*]. We say that F is a *maximal hadwigerian* subgraph of G [we recall that G is uniform], **if** F is a *hadwigerian subgraph* of G and $\omega(F)$ is **maximum** for this property [it is immediate that such a F exists and is well defined].

Now we define the *hadwiger caliber*.

Definition 2.16. (*Hadwiger caliber*). Let G be uniform, and let F be a *maximal hadwigerian* subgraph of G [use Definitions 2.15] , then the *hadwiger caliber* of G is denoted by $a(G)$, where $a(G) = \omega(F)$.

The following remark clearly shows that for every uniform graph G, $a(G)$ exists and is well defined.

Remark 2.17. *For every uniform graph G, the hadwiger caliber $a(G)$ exists and is well defined.*
Proof. Let G be uniform and let F be a **maximal hadwigerian subgraph** of G [use Definitions 2.15]; observing [by definition of a *maximal hadwigerian subgraph* of G] that F is a *hadwigerian subgraph* of G and $\omega(F)$ is **maximum** for this property, clearly $\omega(F)$

is unique and therefore $a(G)$ is also unique, since $a(G) = \omega(F)$. So $a(G)$ exists and is well defined. □

It is immediate that the hadwiger caliber [i.e. the graph parameter a] is defined for *uniform graphs* and only for *uniform graphs*. We will see that the hadwiger caliber plays a crucial role for the algebraic reformulation of the Hadwiger conjecture. Now, using the definition of a uniform graph [use Definition 2.5], the definition of a relative subgraph [use Definition 2.10] and the definition of the hadwiger caliber [use Definition 2.16], then the following assertion becomes immediate.

Assertion 2.18. *Let G be uniform and let $a(G)$ be the hadwiger caliber of G. Consider $\tau(G)$ [$\tau(G)$ is the hadwiger index of G (use Definitions 1.7)]. We have the following six properties.*
(2.18.0). $\omega(G) \geq a(G)$.
(2.18.1). G is hadwigerian $\Leftrightarrow \tau(G) = \omega(G) = a(G) \Leftrightarrow \tau(G) = \omega(G) \Leftrightarrow a(G) = \omega(G)$.
(2.18.2). G is not hadwigerian $\Leftrightarrow \omega(G) > a(G) \Leftrightarrow \omega(G) \neq \tau(G)$.
(2.18.3). If $\omega(G) \in \{0, 1, 2\}$, then $a(G) = \omega(G) = \tau(G)$ (i.e. G is hadwigerian).
(2.18.4). If $\omega(G) \geq j$ (where $j \in \{0, 1, 2\}$), then $a(G) \geq j$.
(2.18.5). For every relative subgraph R of G [use Definition 2.10], we have $a(R) \leq a(G)$.
Proof. Property (2.18.0) is immediate [use the definition of $a(G)$]; properties (2.18.1) and (2.18.2) are trivial [use the definitions of $a(G)$ and $\tau(G)$]. Properties (2.18.3) is easy $\Big($ indeed, let G be uniform such that $\omega(G) = j$ where $j \in \{0, 1, 2\}$, clearly $\chi(G) = j$ where $j \in \{0, 1, 2\}$; observe $\omega(G) = \tau(G)$ [use the previous and property (3) of Assertion 1.9]. Now using the previous equality and property (2.18.1), then it becomes trivial to deduce that $a(G) = \omega(G) = \tau(G)$ and G is hadwigerian $\Big]$. Property (2.18.4) is an immediate consequence of property (2.18.3). Property (2.18.5) immediately results by using the definition of a relative subgraph [use Definition 2.10] and the definition of the parameter a [use Definition 2.16]. □

The previous definitions and simple properties made, now the following Theorem is the algebraic reformulation of the Hadwiger conjecture.

Theorem 2.19. *(The algebraic reformulation of the Hadwiger conjecture). The following are equivalent.*
(i) For every uniform graph U, we have $\omega(U) = a(U)$.
(ii) The Hadwiger conjecture is true [i.e. For every graph G', we have $\chi(G') \leq \eta(G')$].
Proof. $(i) \Rightarrow (ii)$]. Indeed, observe [by the hypotheses] that for every uniform graph U, we have $\omega(U) = a(U)$; now using the previous and property (2.18.1) of Assertion 2.18, then it becomes trivial to deduce that

$$for\, every\, uniform\, graph\, U,\, we\, have \quad \omega(U) = a(U) = \tau(U) \qquad (2.0).$$

Now let G be a graph and let P be uniform such that P is isomorphic to a parent of G [such a P clearly exists via Assertion 2.9], clearly

$$\tau(P) \leq \tau(G) \qquad (2.1)$$

[by observing that in particular P is isomorphic to a parent of G and by using Remark 2.0]. Since in particular P is isomorphic to a parent of G, clearly P is isomorphic to a true pal of G and so

$$\chi(P) = \chi(G) \qquad (2.2).$$

Clearly $\omega(P) = \chi(P)$ [since $P \in \Omega$] and using the previous equality, then it becomes trivial to deduce that equality *(2.2)* clearly says that

$$\omega(P) = \chi(G) \qquad (2.3).$$

Recalling that P is uniform and using *(2.0)*, then it becomes trivial to deduce that

$$\omega(P) = a(P) = \tau(P) \qquad (2.4).$$

Now using *(2.4)* and *(2.3)* and *(2.1)*, then it becomes trivial to deduce that

$$\omega(P) = a(P) = \tau(P) = \chi(G) \text{ and } \tau(P) \leq \tau(G) \qquad (2.5).$$

Using *(2.5)*, then it becomes trivial to deduce that

$$\omega(P) \leq \chi(G) \leq a(P) \leq \tau(P) \leq \tau(G) \qquad (2.6).$$

(2.6) immediately implies that

$$\chi(G) \leq \tau(G) \qquad (2.7).$$

It is trivial that $\tau(G) \leq \eta(G)$ and using the previous inequality, then it becomes trivial to deduce that inequality *(2.7)* clearly says that

$$\chi(G) \leq \tau(G) \leq \eta(G) \qquad (2.8).$$

Clearly $\chi(G) \leq \eta(G)$ [use *(2.8)*] and the previous inequality clearly says that the Hadwiger conjecture is true for G; using the previous and observing that the graph G was arbitrary chosen, then it becomes trivial to deduce that every graph G' satisfies $\chi(G') \leq \eta(G')$; so the Hadwiger conjecture is true and therefore $(i) \Rightarrow (ii)$].

$(ii) \Rightarrow (i)$]. Immediate $\Big($ indeed, if the Hadwiger conjecture is true [i.e. if for every graph G', we have $\chi(G') \leq \eta(G')$], then, using Theorem 1.8 [the original reformulation of the Hadwiger conjecture], we immediately deduce that

$$for\ every\ G \in \Omega,\ we\ have\ \omega(G) = \tau(G) \qquad (2.9).$$

Now let U be uniform; observing that $U \in \Omega$ and using *(2.9)*, then we immediately deduce that

$$for\ every\ uniform\ graph\ U,\ we\ have\ \omega(U) = \tau(U) \qquad (2.10).$$

Now using *(2.10)* and property (2.18.1) of Proposition 2.18, then it becomes trivial to deduce that for every uniform graph U, we have $\omega(U) = a(U)$. So $(ii) \Rightarrow (i)]$ $\Big)$. Theorem 2.19 follows. □

Visibly, the algebraic reformulation of the Hadwiger conjecture given by Theorem 2.19 clearly use the original reformulation of the Hadwiger conjecture and shows that the proof of this conjecture is strongly linked to *uniform* graphs which are a very small class of graphs [use Definition 2.5 for a *uniform* graph].

References

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