European Journal of Science, Innovation and Technology

Vol. 1 No. 3 (2021)

Basic Trigonometric Identity

Dragan Obradovic[1], Lakshmi Narayan Mishra[2*], Vishnu Narayan Mishra[3]
[1]Elementary school “Jovan Cvijic”, Kostolac-Pozarevac, Serbia
[2]Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology (VIT) University, Vellore, Tamil Nadu, India
[3]Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak 484 887, Madhya Pradesh, India

Abstract. Trigonometry is a branch of mathematics that deals with specific functions of angles and their application. Basic trigonometric identities are equations that establish a relationship between the sine, cosine, tangent, and cotangent of one angle, and allow you to find any of these trigonometric functions through a known other.

Keywords: trigonometry, angles, trigonometric functions, trigonometric identity

Introduction

Trigonometry is a mathematical discipline that deals with the study of the relationship between the elements of a right triangle, which express angles with the help of pages and vice versa. The word "trigonometry" is derived from the ancient Greek words trigonom (triangle) and metron (measure). Metric properties are known that connect only the angles or only the sides of a right triangle, the most important of which is the Pythagorean theorem. It allows a third to be calculated based on two known sides of a right triangle. Also, one acute angle can be calculated in a right triangle if another is known. However, if one side of a right triangle and one acute angle were known, then the other two sides could not be calculated using the hitherto known properties related to a right triangle. That is why trigonometry was created.

Students encounter trigonometry for the first time at the end of the first year of high school. At that time, they are just getting acquainted with the most basic concepts related to the mentioned topic, while in the second year, according to the plan and program, much more serious tasks are being done, including trigonometric equations. First, the simplest equations are made, and later they are reduced to more complex ones.

For example, some trigonometric equations require different ideas in solving them. That is one of the reasons why students do not like trigonometry. Although many mathematical problems can be solved in several ways, here they are solved in a simpler way. Many trigonometric formulas were used for that, as well as the theory in this field (Mintakovic & Franic, 1995).

Trigonometric Identity

The basic connection between the trigonometric functions sine and cosine is given by the following proposition (Pavkovic & Veljan, 1995).

Proposition 1. For every $t \in \mathbb{R}$ holds

$$\cos^2 t + \sin^2 t = 1. \tag{1}$$

Proof 1. The coordinates of the point (x, y) on a single circle satisfy the equation $x^2 + y^2 = 1$. How are the coordinates of the point $E(t)$ obtained by exponential mapping of the point $t \in \mathbb{R}$ on a single black circle equal to $\cos t$ and $\sin t$, the stated identity follows.

[*]Corresponding Author: Lakshmi Narayan Mishra (lakshminarayan.mishra@vit.ac.in)

Proof 2. If we understand the trigonometric functions of the proposition as functions of the angle (e.g. $\emptyset$), then $\cos^2 \emptyset + \sin^2 \emptyset = 1$. The proof now follows from Figure 2 analogously as proof 1.

Trigonometric functions are, as we have seen, real functions of a real variable. They are often associated with trigonometric functions of an angle whose values are again from $\mathbb{R}$: Let a unit circle be given in the coordinate system as in Figure 1, and let an angle be given$\emptyset$.

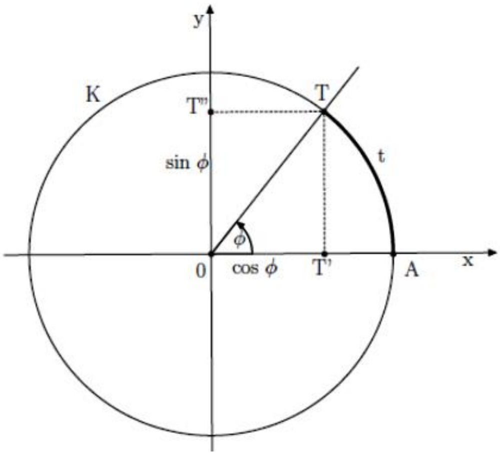

Figure 1. Trigonometric functions of an angle

Proof 3. Consider a right triangle as in Figure 5. In each right the triangle is valid for Pythagoras' teaching $a^2 + b^2 = c^2$. If we divide this equation by c^2 (which is certainly different from 0), follows.

Historically, the first interpretation of trigonometric functions was the interpretation of acute-angle trigonometric functions. In many applications, it is sufficient to observe the above functions only for sharp angles. Every sharp angle is an angle of some rectangle triangle, and we will therefore deny trigonometric functions using it.

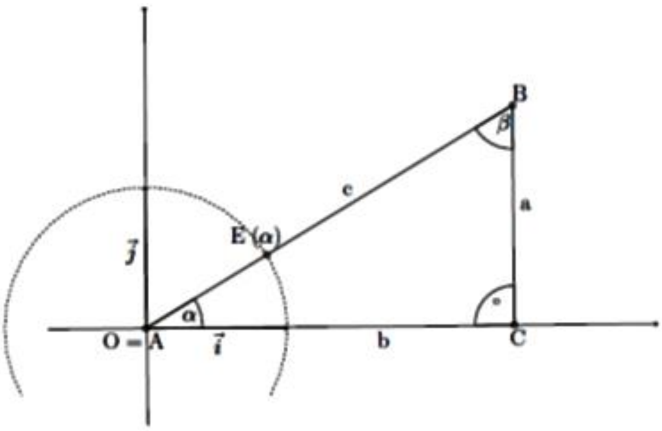

Figure 2. Interpretation using a right triangle

$$\left(\frac{b}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$

How we deranged the trigonometric functions in a right triangle with $\sin \alpha = \frac{a}{c}$ i $\cos \alpha = \frac{a}{c}$, equality follows

$$\cos^2\alpha + \sin^2 \alpha = 1:$$

Another trigonometric identity is

$$\text{tg } t \,.\, \text{ctg } t = 1: \qquad (2)$$

It follows directly from the definitions of the trigonometric functions tangent and cotangent. If the default value of one trigonometric function of an angle, from formulas (1) and (2) the value of the remaining trigonometric functions of that angle can be calculated. Let's express now the remaining trigonometric functions if one of them is given.

1. If the default son *t* ...
It follows from (1) $\cos^2 t = 1 - \sin^2 t$, that is

$$\cos t = \pm\sqrt{1-\sin^2 t}.$$

How is tg $t = \frac{\sin t}{\cos t}$, $a\ ctg\ t = \frac{\cos t}{\sin t}$, follows

$$\text{tg } t = \pm\frac{\sin t}{\sqrt{1-\sin^2 t}},$$

$$\text{ctg } t = \pm\frac{\sqrt{1-\sin^2 t}}{\sin t}.$$

2. If the default cos t ...
From (1) follows $\sin^2 t = 1 - \cos^2 t$, respectively

$$\sin t = \pm\sqrt{1-\cos^2 t}.$$

It follows

$$\text{tg } t = \pm\frac{\sqrt{1-\cos^2 t}}{\cos t},$$

$$\text{ctg } t = \pm\frac{\cos t}{\sqrt{1-\cos^2 t}}.$$

3. If the default tg t ...
If (1) we divide by $\cos^2 t$ we get

$$\frac{1}{\cos^2 t} = \text{tg}^2\, t,$$

whence it follows

$$\cos t = \pm\frac{1}{\sqrt{1+\text{tg}^2\, t}}.$$

If tg $t = \frac{\sin t}{\cos t}$ we write in the form sin t = tg t .cos t, follows

$$\sin t = \pm\frac{\text{tg } t}{\sqrt{1+\text{tg}^2\, t}}.$$

It follows from (2)

$$\text{ctg } t = \frac{1}{\text{tg}^2\, t}.$$

4. If the default ctg *t* ...
If (1) we divide by $\sin^2 t$ we get

$$\frac{1}{\sin^2 t} = 1 + \text{ctg}^2\, t,$$

whence it follows

$$\sin t = \pm\frac{1}{\sqrt{1+\text{ctg}^2\, t}}.$$

If ctg t = $\frac{\cos t}{\sin t}$ we write in the form cos t = ctg .sin *t*, follows

$$\cos t = \pm\frac{\operatorname{ctg} t}{\sqrt{1+\operatorname{ctg}^2 t}}.$$

It follows from (2)

$$\operatorname{tg} t = \frac{1}{\operatorname{ctg}^2 t}.$$

The derived relations are shown in the following table.

Table 1

Default / Required	sin t	cos t	tg t	ctg t
sin t	sin t	$\pm\sqrt{1-\sin^2 t}$	$\pm\frac{\operatorname{tg} t}{\sqrt{1+\operatorname{tg}^2 t}}.$	$\pm\frac{1}{\sqrt{1+\operatorname{ctg}^2 t}}$
cos t	$\pm\sqrt{1-\sin^2 t}$	cos t	$\pm\frac{1}{\sqrt{1+\operatorname{tg}^2 t}}.$	$\pm\frac{\operatorname{ctg} t}{\sqrt{1+\operatorname{ctg}^2 t}}$
tg t	$\pm\frac{\sin t}{\sqrt{1-\sin^2 t}}$	$\pm\frac{\sqrt{1-\cos^2 t}}{\cos t}$	tg t	$\frac{1}{\operatorname{ctg} t}$
ctg t	$\pm\frac{\sqrt{1-\sin^2 t}}{\sin t}$	$\pm\frac{\cos t}{\sqrt{1-\cos^2 t}}$	$\frac{1}{\operatorname{tg} t}$	ctg t

Use of Trigonometric Identities

We carry out the proof in two steps. In the first step, we prove one lemma, and in the second step, based on the lemma, we prove Heron's formula (Miles, 2007).

Lemma. If α + β + γ = π, then it is

$$ctg\frac{\alpha}{2}+ctg\frac{\beta}{2}+ctg\frac{\gamma}{2}=ctg\frac{\alpha}{2}\cdot ctg\frac{\beta}{2}\cdot ctg\frac{\gamma}{2}.$$

In other words, the lemma claims that the sum of the cotangents of the half angles is equal to the product of the cotangents of the half angles.

Proof of the lemma.

From α + β + γ = π it follows that

$$\gamma=\pi-(\alpha+\beta),\ \frac{\gamma}{2}=\frac{\pi}{2}-\frac{\alpha+\beta}{2},\ ctg\frac{\gamma}{2}=tg\frac{\alpha+\beta}{2},$$

so we can write

$$ctg\frac{\alpha}{2}+ctg\frac{\beta}{2}+ctg\frac{\gamma}{2}=ctg\frac{\alpha}{2}+ctg\frac{\beta}{2}+tg\frac{\alpha+\beta}{2}=\frac{1}{tg\frac{\alpha}{2}}+\frac{1}{tg\frac{\beta}{2}}+\frac{tg\frac{\alpha}{2}+tg\frac{\beta}{2}}{1-tg\frac{\alpha}{2}tg\frac{\beta}{2}}. \quad (3)$$

If we rearrange the right side in formula (3), it follows

$$ctg\frac{\alpha}{2}+ctg\frac{\beta}{2}+tg\frac{\alpha+\beta}{2}=\frac{tg\frac{\alpha}{2}+tg\frac{\beta}{2}}{tg\frac{\alpha}{2}\cdot tg\frac{\beta}{2}}+\frac{tg\frac{\alpha}{2}+tg\frac{\beta}{2}}{1-tg\frac{\alpha}{2}tg\frac{\beta}{2}} \quad (4)$$

After adding the terms on the right-hand side of equation (4) we get

$$ctg\frac{\alpha}{2}+ctg\frac{\beta}{2}+tg\frac{\alpha+\beta}{2}=\frac{1}{tg\frac{\alpha}{2}}\cdot\frac{1}{tg\frac{\beta}{2}}\cdot\frac{tg\frac{\alpha}{2}+tg\frac{\beta}{2}}{1-tg\frac{\alpha}{2}tg\frac{\beta}{2}} \quad (5)$$

If we notice that the right side in (5) is the second notation of the formula, we conclude that it is fulfilled

$$ctg\frac{\alpha}{2}+ctg\frac{\beta}{2}+tg\frac{\alpha+\beta}{2}=ctg\frac{\alpha}{2}\cdot ctg\frac{\beta}{2}\cdot ctg\frac{\gamma}{2}$$

whereby the lemma is proved.

Let us return to the proof of the formula. According to Figure 3 (from the definition of the ctg function) we can write

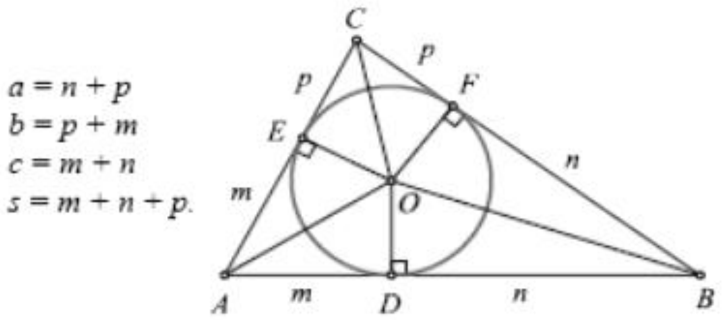

Figure 3

$$ctg\frac{\alpha}{2}+ctg\frac{\beta}{2}+ctg\frac{\gamma}{2}=\frac{p}{r}+\frac{n}{r}+\frac{m}{r}=\frac{s}{r}, \quad (6)$$

while on the other hand,

$$ctg\frac{\alpha}{2}\cdot ctg\frac{\beta}{2}\cdot ctg\frac{\gamma}{2}=\frac{p}{r}\cdot\frac{n}{r}\cdot\frac{m}{r}=\frac{pnm}{r^3}. \quad (7)$$

From formulas (6) and (7) and based on the above lemma, the following holds:

$$r^2s = pnm \Leftrightarrow r^2s2 = spnm \Leftrightarrow P^2 = spnm. \quad (8)$$

If we eliminate p, n, m from (8), taking into account the relations p = s - a, n = s - b, m = s - c, we obtain finally, again according to (8) that

$$P=\sqrt{s(s-a)(s-b)(s-c)}.$$

Instead of a conclusion, we note that evidence requires knowledge of basic trigonometric functions and relations, so it could not be performed at the level of primary school teaching, unlike evidence based on Pythagoras' theorem (Dakic & Elezovic, 2001) and can serve as an illustration, application of the same theorem.

Conclusion

The idea of this paper is to bring trigonometry closer to students, as well as to remove many doubts about certain mathematical problems from one of the more difficult topics in mathematics that are taught in school. For that, different examples were chosen in which many ideas are used. Some can be used in additional classes, to prepare for competitions or admissions. The problems are gradually solved and should indicate the method of solving problems from trigonometry, so that we reduce complicated mathematical problems to simple ones in different ways.

Acknowledgment

The authors are very much thankful to anonymous reviewers for their useful comments which improved the quality of this research article.

References

Dakic, B. & Elezovic, N. (2001). *Mathematics 1* [Textbook and collection of tasks, for the 1st grade of Element Elementary School].

Miles, D. E. (2007). FA Proof of Heron's Formula. *American Mathematics Monthly*, *114*(10), 937.

Mintakovic, S. & Franic, M. (1995). *Trigonometry* [High school exercise book]. Element, Zagreb.

Pavkovic, B. & Veljan, D. (1995). *Elementary Mathematics 2* [Schoolbook]. Zagreb.